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POWER ITERATIONS AND THE DOMINANT EIGENVALUE PROBLEM

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POWER ITERATIONS AND THE DOMINANT EIGENVALUE PROBLEM

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Abstract. The orbits of an iterative numerical method for the dominant eigenvalue problem are analyzed from a discrete dynamical systems perspective. It is shown that the method can extract more information than the standard power method but at greater computational cost.

Key words. Power iteration, Power method

1. Introduction

The *power iteration* is the matrix iteration

$$V_{n+1} = B*V_n/\|V_n\| \tag{1.1}$$

where V_0 is a given nonzero m-vector, B is a real m×m matrix, and $\|\cdot\|$ is the Euclidean vector norm [10,11]. It is similar to the *power method* for finding the dominant eigenvalue of a real matrix,

$$y_{n+1} = A*v_n \quad , \quad v_{n+1} = y_{n+1}/\mu_n \tag{1.2}$$

where A is a real matrix with a dominant eigenvalue, v_0 is an initial estimate of an eigenvector associated with the dominant eigenvalue of A, and μ_n is an element of y_{n+1} with the property that

$$|\mu_n| = \|y_{n+1}\|_\infty$$

(see [1, p.144]). We will show that although the power iteration (1.1) is generally slower than the power method (1.2), it can provide extra information about the dominant eigenvalue(s) of a matrix in certain cases. We take a geometric approach, viewing (1.1) as a discrete dynamical system and inquiring as to the nature of its limit sets (attractors) in various cases (in the spirit of [9]).

The iteration (1.1) is considered in a different context as a special case of the $\mathbb{R}^n \to \mathbb{R}^n$ map

$$V_{n+1} = A*V_n + B*V_n/\|V_n\| \tag{1.3}$$

in [10], based on work in [2] (also reported in [3]). Further details on the iteration (1.3) may be found in [10,12,13] and the forthcoming [4]. Although (1.3) is only a linear perturbation of the well-behaved iteration (1.1), it exhibits strange attractors and apparently chaotic dynamics. Of course, (1.1) can also be viewed as (1.2) with a change of normalization, and much is known about the numerical method given by (1.2) (see also [8, p.352]).

2. The Power Method

The power method (1.2) has the property that if A is a nondefective matrix with a dominant eigenvalue, say λ_1, and v_0 has a nonzero projection on an eigenvector associated with this dominant eigenvalue, then

$$\mu_n \longrightarrow \lambda_1 \quad \text{as} \quad n \longrightarrow \infty$$

and v_n converges to an eigenvector associated with λ_1 and with unit ℓ_∞ norm. If v_0 does not have a nonzero component along an eigenvector associated with λ_1 and infinite precision arithmetic is used then we must consider the eigenvalue of largest modulus along which v_0 does have a nonzero component. In actual computations, however, a component along an eigenvector associated with λ_1 would almost certainly be introduced eventually and magnified in successive iterations [1, p.148]. The same results are found if A is defective (considering now principal vectors [7, p.3] rather than just eigenvectors) but the convergence is much slower.

When the dominant eigenvalue is real, the method converges to a fixed point. However, when the dominant eigenvalue is a complex conjugate pair, the method generally fails to converge. Methods exist to recover information in such cases [6, p.257] but they tend to be somewhat involved.

When A and v_0 are real, up to a change of sign we have

$$v_{n+1} = A*y_n/\|y_n\|_\infty \quad , \quad v_{n+1} = A*y_{n+1}/\|A*y_n\|_\infty$$

and in this formulation v_n need not be calculated until it is actually needed (to estimate λ_1). Then the iteration for y_{n+1} is the same as (1.1) except for the particular ℓ_p norm used in the normalization. For this reason we sometimes refer to the quantity μ_n in (1.2) as the *signed* ℓ_∞ norm.

3. Conic Orbits

If B is nonsingular then the points V_1, V_2, V_3.... of the orbit of (1.1) all lie on a conic defined by the matrix

$$G = (B*B^T)^{-1}$$

i.e.

$$V_i^T*G*V_i = 1$$

for all $i \geq 1$. For,

$$\begin{aligned} V_{n+1}^T*G*V_{n+1} &= (V_n^T*B^T/\|V_n\|)*G*(B*V_n/\|V_n\|) \\ &= V_n^T*(B^T*G*B)*V_n/\|V_n\|^2 \\ &= V_n^T*I*V_n/\|V_n\|^2 \\ &= 1 \end{aligned}$$

for every $n \geq 1$ and for any V_0 which is nonzero. Clearly G is positive definite symmetric, and so the points V_1, V_2, V_3,... must all lie on the hyperellipse defined by

$$V^T*G*V = 1 \qquad (3.1)$$

in $\mathbb{R}^m$.

If B is singular, a similar result holds. In order to handle simultaneously both the case where B is simple and the case where B is defective we state the result in terms of principal vectors. We have the following theorem:

Theorem 1: Suppose B is a real square m×m matrix with $0 \le q \le m$ null eigenvalues. If $q>0$ and V_0 has a nonzero component along a principal vector associated with a nonzero eigenvalue, then all orbits of the power iteration (1.1) are constrained to a hyperellipse in (m-q)-dimensions (for all but finitely many n). Otherwise, the orbit reaches the origin in finitely many iterations.

Proof: First, note that

$$\begin{aligned} V_1 &= B*V_0/\|V_0\| \\ V_2 &= B*V_1/\|V_1\| \\ &= B*(B*V_0/\|V_0\|)/\|\, B*V_0/\|V_0\|\, \| \\ &= B^2*V_0/\|B*V_0\| \end{aligned}$$

and, in general,

$$V_n = B^n*V_0/\|B^{n-1}*V_0\| \tag{3.2}$$

for $n \ge 1$. Now let

$$J = R^{-1}*B*R$$

be the Jordan normal form of B for some nonsingular R. Substituting this into (3.2) gives

$$\begin{aligned} V_n &= (R*J*R^{-1})^n*V_0/\|B^{n-1}*V_0\| \\ &= R*J^n*R^{-1}*V_0/\|B^{n-1}*V_0\| \\ &= \alpha_n R*\left[J^n*Z_0\right] \end{aligned} \tag{3.3}$$

where $Z_0=R^{-1}*V_0$ and

$$\alpha_n = 1/\|B^{n-1}*V_0\|$$

is a scalar (for each n). Clearly, if Z_0 has no nonzero component along a principal vector of J (equivalently, if V_0 has no nonzero component along a principal vector of B) that is associated with a non-null eigenvalue, the term J^n*Z_0 in

(3.3) must eventually become the zero vector. Thus α_{n+1} is undefined and the iteration stops. Otherwise, for n sufficiently large (it suffices that n≥m), all Jordan blocks in J associated with a null eigenvalue will have become blocks of entirely zeros in J^n (since these Jordan blocks are nilpotent). Now, the principal vectors belonging to a given Jordan block do not interact with the remaining principal vectors, in the sense that if v_j is a principal vector associated with $J_i(\lambda)$, a Jordan block of J, then $J^n * v_j$ involves only a linear combination of principal vectors of J that are also associated with $J_i(\lambda)$. Therefore for n large enough that all nilpotent Jordan blocks have become entirely zero submatrices, the vector

$$J^n * Z_0$$

can be written in terms of a basis consisting of only the remaining (m-q) principal vectors. Thus the iteration lies in a (m-q)-dimensional subspace of $\mathbb{R}^m$, and a suitable change of variables can then be used to transform the iteration into one of the form

$$W_{n+1} = C * W_n / \|W_n\|$$

where W_n is a (m-q)-vector for every n and C is a real nonsingular (m-q)x(m-q) matrix. Hence, in this subspace, the iteration is constrained to the hyperellipse determined by the matrix

$$(C * C^T)^{-1}$$

(as was shown previously for the nonsingular case) for all but finitely many n. □

We emphasize that this is not an asymptotic result; after a bounded number of steps, the points lie precisely on the hyperellipse (assuming infinite precision). We now wish to look at the orbits on the attracting hyperellipses.

4. Limit Orbits for Nondefective Matrices

Suppose that B is nondefective and nonsingular, and let $x_1, \ldots, x_m$ be a set of eigenvectors associated with the eigenvalues $\lambda_1, \ldots, \lambda_m$, respectively, with

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_m|.$$

Then any $V_0 \in \mathbb{R}^m$ may be written as

$$V_0 = \sum_{i=1}^{m} \alpha_i x_i$$

where α_i is the component of V_0 along x_i. Computing the power iteration (1.1) by (3.2) gives

$$V_n = \left[\sum_{i=1}^{m} \alpha_i \lambda_i^n x_i\right] \div \left\| \sum_{i=1}^{m} \alpha_i \lambda_i^{n-1} x_i \right\| \qquad (4.1)$$

for $n \geq 1$.

Now suppose that B has a (repeated) dominant eigenvalue λ_1 with multiplicity $1 \leq r \leq m$, that is,

$$\lambda_1 = \lambda_2 = \cdots = \lambda_r$$

and

$$|\lambda_1| > |\lambda_{r+1}| \geq \cdots \geq |\lambda_m|$$

Removing a factor of λ_1^n from the numerator and λ_1^{n-1} from the denominator of (4.1) gives

$$V_n = \lambda_1^n \cdot \left[\alpha_1 x_1 + \cdots + \alpha_r x_r + \mathcal{O}(\lambda_{r+1}/\lambda_1)^n\right] \div$$
$$\lambda_1^{n-1} \cdot \left\|\alpha_1 x_1 + \cdots + \alpha_r x_r + \mathcal{O}(\lambda_{r+1}/\lambda_1)^{n-1}\right\|$$

and clearly, in the limit

$$V_n \longrightarrow \gamma_n x$$

where x is a unit vector in span$\{x_1, \ldots, x_r\}$, and

$$\gamma_n = \lambda_1 \cdot (\lambda_1 / |\lambda_1|)^{n-1} \qquad (4.2)$$
$$= (\mathrm{sgn}(\lambda_1))^{n-1} \cdot \lambda_1$$

(provided that at least one of $\alpha_1, \ldots, \alpha_r$ is nonzero; otherwise, we begin the analysis anew by considering λ_{r+1}). Hence if $\lambda_1 > 0$ the iteration tends to a fixed point, and if $\lambda_1 < 0$ the iteration tends to a symmetric (in the origin) period two cycle on the points $\pm\lambda_1 x$.

Now suppose that B has a real positive eigenvalue λ_1 of multiplicity p, so that

$$\lambda_1 = \lambda_2 = \cdots = \lambda_p$$

and a real negative eigenvalue $-\lambda_1$ of multiplicity q, so that

$$\lambda_{p+1} = \lambda_{p+2} = \cdots = \lambda_{p+q}$$

and suppose further that

$$|\lambda_1| > |\lambda_{p+q+1}| \geq \cdots \geq |\lambda_m|$$

Proceeding as before, we have that

$$V_n = \gamma_n \cdot \left(\sum_{i=1}^{p} \alpha_i x_i \pm \sum_{i=p+1}^{p+q} \alpha_i x_i + O(\lambda_{p+q+1}/\lambda_1)^n \right) \div \left\| \sum_{i=1}^{p} \alpha_i x_i \mp \sum_{i=p+1}^{p+q} \alpha_i x_i + O(\lambda_{p+q+1}/\lambda_1)^{n-1} \right\|$$

(with γ_n as given in (4.2)) where the $\pm$ is positive when n is even and negative when n is odd in the numerator, and contrarily in the denominator. In the limit, we have that (approximately)

$$V_n = \gamma_n \cdot (y_1 + y_2) / \|y_1 - y_2\|$$
$$V_{n+1} = \gamma_{n+1} \cdot (y_1 - y_2) / \|y_1 + y_2\|$$

where

$$y_1 = \sum_{i=1}^{p} \alpha_i x_i$$
$$y_2 = \sum_{i=p+1}^{p+q} \alpha_i x_i$$

Since λ_1 is positive, $\gamma_n = \lambda_1$ for all n, and so we have a period two orbit

$$V_n = \lambda_1 \cdot (y_1 + y_2)/\|y_1 - y_2\|$$
$$V_{n+1} = \lambda_1 \cdot (y_1 - y_2)/\|y_1 + y_2\|$$

which in general is not a symmetric orbit (in the origin). We will not in general have $\|V_n\|=\lambda_1$ in this case, but note that

$$\|V_n\| = |\lambda_1| \cdot \|y_1 + y_2\|/\|y_1 - y_2\|$$
$$\|V_{n+1}\| = |\lambda_1| \cdot \|y_1 - y_2\|/\|y_1 + y_2\|$$

so that

$$\|V_n\| \cdot \|V_{n+1}\| = |\lambda_1|^2$$

in the limit of large n; the modulus of the dominant eigenvalue is the geometric mean of the norms of two successive iterates (in the limit). Note that the asymmetry of the orbit allows us to distinguish this case from the case where λ_1 is a negative dominant eigenvalue (which always gives a symmetric limiting orbit) most of the time (i.e. when the resulting asymptotic orbit is indeed asymmetric).

Now let us suppose that B has a complex conjugate pair of eigenvalues that is dominant, i.e. that λ_1 and λ_2 are complex conjugates, and

$$|\lambda_1| > |\lambda_3| \geq \cdots \geq |\lambda_m|$$

From (4.1) we have

$$V_n = |\lambda_1| \cdot \left(\alpha_1 e^{\iota n\theta} x_1 + \alpha_2 e^{-\iota n\theta} x_2 + \mathcal{O}(\lambda_3/\lambda_1)^n\right) \div \left\|\alpha_1 e^{\iota(n-1)\theta} x_1 + \alpha_2 e^{-\iota(n-1)\theta} x_2 + \mathcal{O}(\lambda_3/\lambda_1)^{n-1}\right\|$$

where $\theta=\arg(\lambda_1)$. In the limit, this becomes

$$V_n = |\lambda_1| \cdot \left(\alpha_1 e^{\iota n\theta} x_1 + \alpha_2 e^{-\iota n\theta} x_2\right) \div \left\|\alpha_1 e^{\iota(n-1)\theta} x_1 + \alpha_2 e^{-\iota(n-1)\theta} x_2\right\| \qquad (4.3)$$

Thus if θ is such that $\exp(\iota\theta)=\exp(\iota(n+1)\theta)$ for some n (i.e. if λ_1^n and $\bar{\lambda}_1^n$ are both in $\mathbb{R}^+$ for some n) then we have an asymptotically period n orbit for the sequence $\{V_n\}$, otherwise the orbit is aperiodic on the underlying hyperellipse. If the orbit is period n then we note that the geometric mean of the norms of n consecutive iterates tends to $|\lambda_1|$, as is easily seen by writing out the product of n iterates and noting the cancellation.

In a similar way, if B has two pairs of complex conjugate eigenvalues of equal modulus and all other eigenvalues of B have lesser moduli, then we have a situation like that of (4.3) save that there is an additional angle to be considered. Hence if the first pair alone would give a periodic orbit of period p, and the second pair alone would give a periodic orbit of period q, then the orbit of the iteration will be periodic with period n=lcm(p,q). If either pair alone would give an aperiodic orbit, then the orbit is aperiodic. Again the geometric mean of n consecutive iterates tends to $|\lambda_1|$ when the orbit is asymptotically period n. The obvious generalization holds for more than two complex conjugate pairs of equal moduli. In particular, in an even-dimensional space, say of dimension 2r, choosing B to be a 2r×2r matrix with r complex conjugate pairs of eigenvalues, all of equal modulus and such that

$$\lambda_i^n \notin \mathbb{R}$$

for all i=1,...,2r and n=1,2,3,... gives a method for generating a sequence of points on the ellipse given by (3.1) by iterating (1.1) with an arbitrary nonzero V_0.

If in addition to some number of complex conjugate pairs of eigenvalues with equal modulus there are some number of positive and/or negative real eigenvalues of the same modulus, then it is immediate that the iteration is aperiodic if any complex conjugate eigenvalue gives an aperiodic orbit, and periodic otherwise, with period given by the least common multiple of the individual complex conjugate pairs and the period two due to the negative real eigenvalues, if present.

We have established the following theorem:

Theorem 2: Consider the iteration

$$V_{n+1} = B*V_n/\|V_n\|$$

where B is a real mxm nondefective matrix and V_0 is a given nonzero m-vector. Suppose that V_0 has a nonzero component along an eigenvector of B which is associated with an eigenvalue of B with maximum modulus. Then all iterates for $n \geq m$ are constrained to an ellipse in a subspace of $\mathbb{R}^m$ of dimension rank(B), and the asymptotic behaviour of the iterates is as follows:

i) if B has a multiple real dominant eigenvalue, then the iteration tends to a fixed point;

ii) if B has a multiple negative real eigenvalue, then the iteration tends to a symmetric period two cycle;

iii) if B has both positive and negative real dominant eigenvalues, then the iteration tends to a period two cycle;

iv) if B has a complex conjugate dominant eigenvalue, then the iteration tends to a period n cycle if the eigenvalues are a multiple of an n<u>th</u> root of unity and is aperiodic otherwise;

v) if B has multiple complex conjugate dominant eigenvalues and multiple real dominant eigenvalues, then the iteration tends to a periodic orbit with period equal to the least common multiple of the periods associated with the individual eigenvalues when all eigenvalues give rise to periodic orbits, and is aperiodic otherwise.

5. Limit Orbits for Defective Matrices

The above results are essentially unchanged if B is defective. Let B be defective and let x be an eigenvector associated with a dominant eigenvalue λ. Let y be a linear combination of principal vectors of B associated with λ. Then the iteration

$$B^n y$$

converges to x as O(1/n) [10, p.88], [16, p.582]. Since the principal vectors corresponding to distinct eigenvalues are noninteracting, it is clear that the qualitative results of Theorem 2 are unchanged, although the convergence to the asymptotic orbits may be exceedingly slow. Hence Theorem 2 remains valid if we allow B to be defective, and require that V_0 have a nonzero projection on a principal vector of B associated with an eigenvalue of maximum modulus. If $J=R^{-1}*B*R$ is the Jordan normal form of B with the dominant eigenvalue (of multiplicity p) in the first block(s) along the diagonal, then these vectors have the form

$$R*e_i$$

$(i=1,\dots,p)$, where $\{e_i\}_{i=1}^{m}$ is the natural basis for $\mathbb{R}^m$.

6. The Power Iteration

The iteration (1.1) in conjunction with Theorem 2 provides a method for the numerical determination of the modulus of the dominant eigenvalue of a real matrix when the resulting orbit is periodic. If the orbit has period $n \geq 1$, then the modulus is approximately the geometric mean of the Euclidean norms of n consecutive iterates. Additionally, the argument is such that the n<u>th</u> power of both the eigenvalue and it's conjugate are positive real numbers, so that the desired eigenvalue λ is given by

$$\lambda = |\lambda|\omega_i \tag{6.1}$$

for some $i=1,\ldots,n$, where $\omega_1,\ldots,\omega_n$ are the n<u>th</u> roots of unity. If the orbit is aperiodic, the dominant eigenvalues are complex conjugates and fail to satisfy (6.1).

As an example of the use of the power iteration, consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \tag{6.2}$$

which has eigenvalues $\pm\sqrt{2}$. In [14 p.98] it is shown that the power method (1.2) applied to this iteration (with initial vector $(1,1)^T$) settles into a period two cycle on the two vectors

$$x_1 = \begin{pmatrix} .5 \\ 1 \end{pmatrix} \quad , \quad x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

i.e. the method fails. The power iteration (1.1) applied to this matrix yields asymptotically the period two cycle (from the same initial vector)

$$x_1 = \sqrt{2}\cdot\begin{pmatrix} .5 \\ 1 \end{pmatrix} \quad , \quad x_2 = (2/\sqrt{5})\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Since this is an asymmetric period two cycle, from Theorem 2 it follows that there are both positive and negative dominant eigenvalues, and that the modulus of these eigenvalues is

$$|\lambda| = (\|x_1\| \cdot \|x_2\|)^{1/2}$$
$$= [(\sqrt{5}/\sqrt{2}) \cdot (2\sqrt{2}/\sqrt{5})]^{1/2}$$
$$= \sqrt{2}$$

Hence the eigenvalues of A are $\pm\sqrt{2}$, as expected. Similar results would be obtained using any norm in place of the Euclidean norm in (1.1).

7. Conclusions

The power iteration is certainly slower than the power method (due to the need to calculate a Euclidean vector norm rather than simply locating an element of the vector with maximum modulus), and additional information can be gained in only a restricted set of cases (primarily when the dominant eigenvalues are complex conjugates and real multiples of a root of unity). Nonetheless, in these cases it does provide useful information about the eigenvalues, and in more general cases i may provide some insight as well. For example, inspection of the elliptical orbits (Theorem 1) can be used to provide information about the existence and multiplicity of null eigenvalues. For these reasons the power iteration may be useful in certain circumstances. Of course, the power iteration/power method is in some sense the basis of most iterative methods for the eigenvalue problem [15] and so this analysis may be useful in the analysis of more practical algorithms for this problem.

We mention that computergraphical evidence seems to indicate that the unsigned power method

$$y_{n+1} = A*y_n / \|y_n\|_\infty \qquad (7.1)$$

has the property that all iterates lie on a ℓ_∞ conic (with respect to some rotation of the axes) asymptotically. As noted in [8, p.362], however, the use of the Euclidean norm in (1.1) greatly facilitates the analysis of the power iteration (see also [5, p.351]) and we have been unable to show a corresponding result for the unsigned power method (7.1). The analysis of cases in Theorem 2 did not depend on which norm was used in (1.1) and as such it holds for (7.1) as well; therefore the comments in §6, excepting those about the ℓ_2 conic orbits, are equally applicable to the unsigned power method, which is no more costly than the usual power method (requiring only that an extra absolute value be taken--but this would be done during the corresponding search for μ_n). Although using (7.1) in place of (1.2) means that the algorithm will not converge (in the usual sense) in the case of a dominant negative real eigenvalue, the method, properly interpreted (see also [6, p.257]), would avoid some of the problems encountered when using the power method. For the matrix A given in (6.2), the unsigned power method (7.1), with initial vector $(1,1)^T$, gives a period two cycle on the vectors

$$x_1 = \begin{pmatrix}1\\2\end{pmatrix} \quad , \quad x_2 = \begin{pmatrix}1\\1\end{pmatrix}$$

and again the fact that this is an asymmetric period two orbit implies that the eigenvalues of A are $\pm|\lambda|$, where

$$\begin{aligned}|\lambda| &= (\|x_1\|_\infty \cdot \|x_2\|_\infty)^{1/2}\\ &= (2\cdot 1)^{1/2}\\ &= \sqrt{2}\end{aligned}$$

as expected. This type of reasoning could easily be incorporated into a standard power method routine.

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